ITERATIVE RECONSTRUCTION METHODS FOR NON-CARTESIAN MRI

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ABSTRACT

For magnetic resonance imaging (MRI) with Cartesian k-space sampling, a simple inverse FFT usually suffices for image reconstruction. More sophisticated image reconstruction methods are needed for non-Cartesian k-space acquisitions. Regularized least-squares methods for image reconstruction involve minimizing a cost function consisting of a least-squares data fit term plus a regularizing roughness penalty that controls noise in the image estimate. Iterative algorithms are usually used to minimize such cost functions. This paper summarizes the formulation of iterative methods for image reconstruction from non-Cartesian k-space samples, and describes some of the benefits of iterative methods. The primary disadvantage of iterative methods is the increased computation time, and methods for accelerating convergence are also discussed.

1. INTRODUCTION

For simplicity, in this summary we consider conventional slice-selective 2D MRI. The extension to 3D is straightforward.

The goal in MR image reconstruction is to estimate the transverse magnetization $f(\vec{r})$ of an object from a finite set of M noisy data samples:

$$y_i = F(\vec{\nu}_i) + \varepsilon_i, \qquad i = 1, \ldots, M, \tag{1}$$

where $F(\vec{\nu})$ denotes the Fourier transform of f for $\vec{\nu} \in \mathbb{R}^2$, defined as follows:

$$F(\vec{\nu}) = \int_{\mathbb{R}^2} f(\vec{r})\, \mathrm{e}^{-\imath 2\pi \vec{\nu} \cdot \vec{r}}\, \mathrm{d}\vec{r}. \tag{2}$$

For simplicity, we ignore field inhomogeneity effects; extensions are available, *e.g.*, [1, 2]. This problem is ill-posed because there are a multitude of continuous-space objects f that exactly match the measured data $\boldsymbol{y} = (y_1, \ldots, y_M)$. So in some sense *all* MRI data is incomplete, and the notion of "partial k-space sampling" is only a matter of degrees of partialness.

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For ill-posed problems involving a continuous-space unknown function f but a finite-dimensional (discrete) data vector $\boldsymbol{y}$, estimation methods can be categorized into three families of solutions. None of these formulations can be called uniquely optimal.

1. **Continuous-continuous formulations**

   In these methods, one first imagines that one has a continuum of measurements, solves the inverse problem under that hypothetical scenario, and then discretizes the solution so that only the available measurements are used.

   In MR, the natural hypothetical set of measurements is the entire Fourier transform $F(\vec{\nu})$ of the object $f(\vec{r})$. If $F(\vec{\nu})$ were available, the "reconstruction method" would be simply an inverse Fourier transform:

   $$f(\vec{r}) = \int_{\mathbb{R}^2} F(\vec{\nu})\, \mathrm{e}^{\imath 2\pi \vec{\nu} \cdot \vec{r}}\, \mathrm{d}\vec{\nu}\,.$$

   For practical implementation one must discretize this "solution" and replace $F(\vec{\nu}_i)$ by the corresponding noisy measurement y_i as follows:

   $$\hat{f}(\vec{r}) = \sum_{i=1}^{M} y_i\, \mathrm{e}^{\imath 2\pi \vec{\nu}_i \cdot \vec{r}}\, w_i, \tag{3}$$

   where $\{w_i : i = 1, \ldots, M\}$ are sampling density compensation factors. This is the conjugate phase method for image reconstruction. Numerous methods have been proposed for choosing the w_i factors, including iterative methods, *e.g.*, [3, 4].

2. **Continuous-discrete formulations**

   These methods attempt to formulate the problem directly using the continuous-discrete model (1). Because there are a multitude of possible solutions, a typical approach is to choose, among those solutions that satisfy (1) exactly, the $\hat{f}$ that has minimum norm, *e.g.*, [5]. However, one may question the appropriateness of insisting on satisfying (1) exactly given the presence of noise. And one may wonder if "minimum norm" is the best criterion for choosing one MR image over other possibilities.

The minimum norm solution turns out to have the form

$$\hat{f}(\vec{r}) = \sum_{i=1}^{M} c_i \, \mathrm{e}^{\imath 2\pi \vec{\nu}_i \cdot \vec{r}}, \qquad (4)$$

where the coefficients $\{c_i : i = 1, \ldots, M\}$ are found by solving a $N \times M$ system of linear equations. In other words, the minimum norm estimate is a linear combination of complex exponentials. The frequencies of those exponentials are those of the k-space sampling. One may wonder if this set of complex exponentials is the most appealing basis for representing f.

3. **Discrete-discrete formulations**

   In these methods, we discretize f using a finite-series expansion akin to (4) but with different basis functions. We focus on this type of formulation hereafter.

1.1. Finite-series object model

For a finite-series approach, we first select some *basis* functions $\{b_j(\vec{r}) : j = 1, \ldots, N\}$ and model the object f as follows:

$$f(\vec{r}) \approx \sum_{j=1}^{N} x_j \, b_j(\vec{r}). \qquad (5)$$

After adopting such a model, the reconstruction problem simplifies to determining the vector of unknown coefficients $\boldsymbol{x} = (x_1, \ldots, x_N)$ from the measurement vector $\boldsymbol{y}$.

Under this assumption, the discrete data, continuous object model (1) simplifies to the following discrete-discrete model:

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\varepsilon}, \qquad (6)$$

where the $M \times N$ *system matrix* $\boldsymbol{A}$ has elements

$$a_{ij} = \int b_j(\vec{r}) \, \mathrm{e}^{-\imath 2\pi \vec{\nu}_i \cdot \vec{r}} \, \mathrm{d}\vec{r}, \qquad (7)$$

for $i = 1, \ldots, M$ and $j = 1, \ldots, N$.

Usually the basis functions in (5) are chosen to be equally-spaced translates of a pulse-like function such as a rectangle or triangle, although more complicated choices such as *prolate spheroidal wave functions* have also been used [6]. Specifically, usually we have

$$b_j(\vec{r}) = b(\vec{r} - \vec{r}_j), \qquad j = 1, \ldots, N, \qquad (8)$$

where $b(\vec{r})$ denotes the common function that is translated to $\vec{r}_j$, the center of the jth basis function.

For basis functions of the form (8), by the shift property of the Fourier transform, the elements of $\boldsymbol{A}$ in (7) are simply

$$a_{ij} = B(\vec{\nu}_i) \, \mathrm{e}^{-\imath 2\pi \vec{\nu}_i \cdot \vec{r}_j}, \qquad (9)$$

where $B(\vec{\nu})$ is the 2-dimensional Fourier transform of $b(\vec{r})$. In other words, the system matrix $\boldsymbol{A}$ has the following form:

$$\boldsymbol{A} = \boldsymbol{B}\boldsymbol{E}, \qquad (10)$$

where $\boldsymbol{B}$ is a $M \times M$ diagonal matrix: $\boldsymbol{B} = \mathsf{diag}\{B(\vec{\nu}_i)\}$, and $\boldsymbol{E} \in \mathbb{C}^{M \times N}$ has elements $E_{ij} = \mathrm{e}^{-\imath 2\pi \vec{\nu}_i \cdot \vec{r}_j}$. In MRI, the matrix $\boldsymbol{E}$ is sometimes called the *Fourier encoding* matrix.

There are numerous possible choices of basis functions $b(\vec{r})$ that have been used in various image reconstruction problems. Typically we simply use rect functions for simplicity, corresponding to square pixels.

1.2. Least-squares reconstruction

Because the measurement noise in MRI is well modeled as complex white gaussian noise, based on the model (6) it may be tempting to apply a least-squares (LS) estimation method:

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\|^2 = (\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{y}.$$

Indeed, for appropriate Cartesian sampling the matrix $\boldsymbol{E}$ in (10) is orthogonal and satisfies $\boldsymbol{E}^{-1} = \frac{1}{M}\boldsymbol{E}^*$ and in this case the LS solution simplifies to $\hat{\boldsymbol{x}} = \boldsymbol{A}^{-1}\boldsymbol{y} = \frac{1}{M}\boldsymbol{E}^*\boldsymbol{B}^{-1}\boldsymbol{y}$, which is essentially the conventional inverse FFT approach. However, for non-Cartesian sampling, the matrix $\boldsymbol{A}$ is often ill-conditioned or even singular, so the LS solution leads to undesirable noise amplification.

1.3. Regularized least-squares methods

To control the noise of the LS estimator, one can modify the cost function by including a regularization term:

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \Psi(\boldsymbol{x})$$

$$\Psi(\boldsymbol{x}) = \frac{1}{2} \|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\|^2_{\boldsymbol{W}} + \beta \, \mathsf{R}(\boldsymbol{x}), \qquad (11)$$

where $\boldsymbol{W}$ is an optional weighting matrix discussed in more detail below. The regularizer $\mathsf{R}(\boldsymbol{x})$ usually penalizes image roughness, and the regularization parameter β controls the tradeoff between spatial resolution and noise. For example, in 1D we might use the squared differences between neighboring pixels:

$$\mathsf{R}(\boldsymbol{x}) = \sum_{j=2}^{N} \frac{1}{2} (x_j - x_{j-1})^2.$$

Nonquadratic regularization is also used in ill-posed inverse problems to better preserve edges. However, in non-Cartesian MRI, often the sampling is good enough that $\boldsymbol{A}$ is only somewhat poorly conditioned so relatively small values of β can be used in which case quadratic regularization may be adequate. A general form for a quadratic regularizer is

$$\mathsf{R}(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}'\boldsymbol{R}\boldsymbol{x},$$

where $\boldsymbol{R}$ is the Hessian of the regularizer. For quadratic regularization, the minimization problem (11) has trhe following explicit solution:

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \Psi(\boldsymbol{x}) = [\boldsymbol{A}'\boldsymbol{W}\boldsymbol{A} + \boldsymbol{R}]^{-1}\boldsymbol{A}'\boldsymbol{W}\boldsymbol{y}. \qquad (12)$$

However, the matrix inverse in this expression is very large ($N \times N$), so in practice one usually computes $\hat{\boldsymbol{x}}$ by using an iteration like the *conjugate gradient* algorithm to minimize the cost function (11), as described below. And if $\mathsf{R}(\boldsymbol{x})$ is nonquadratic, there is no explicit expression for $\hat{\boldsymbol{x}}$, so iterative methods are essential for computing $\hat{\boldsymbol{x}}$.

1.4. Choosing the regularization parameter

A common concern with regularized methods like (11) is choosing the regularization parameter β. Based on (12) we can analyze the spatial resolution properties of $\hat{\boldsymbol{x}}$ easily:

$$\mathsf{E}[\hat{\boldsymbol{x}}] = [\boldsymbol{A}'\boldsymbol{W}\boldsymbol{A} + \beta\boldsymbol{R}]^{-1}\boldsymbol{A}'\boldsymbol{W}\boldsymbol{A}\boldsymbol{x}.$$

Usually the matrix $\boldsymbol{A}'\boldsymbol{W}\boldsymbol{A}$ and the matrix $\boldsymbol{R}$ are Toeplitz, so we can use FFTs to evaluate rapidly the PSF of this image reconstruction method as a function of β. One can then vary β and choose the value that yields the desired FWHM of the local PSF [7].

2. ALGORITHM ACCELERATION

Because the data-fit term in (11) is quadratic, a natural iterative minimization algorithm is the preconditioned *conjugate gradient* (PCG) algorithm. The key step in any gradient-based descent algorithm such as PCG is computing the gradient of $\Psi(\boldsymbol{x})$, which has the form

$$\nabla\Psi(\boldsymbol{x}) = -\boldsymbol{A}'\boldsymbol{W}(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}) + \nabla\,\mathsf{R}(\boldsymbol{x})\,. \qquad (13)$$

The computational bottlenecks are computing the matrix-vector multiplication $\boldsymbol{A}\boldsymbol{x}$ and its transpose $\boldsymbol{A}'\boldsymbol{v}$, without storing $\boldsymbol{A}$ or $\boldsymbol{A}'$ explicitly.

Fortunately, there are efficient and very accurate algorithms for computing these matrix-vector multiplications by using nonuniform *fast Fourier transform* (*NUFFT*) approximations [1,8]. Specifically, each multiplication by $\boldsymbol{A}$ or $\boldsymbol{A}'$ requires an over-sampled FFT and some simple interpolation operations. (This operation is akin to "reverse gridding," *e.g.,* [9–11].) One can precompute and store the interpolation coefficients or compute them as needed [8]. Particularly efficient methods are available for gaussian interpolation kernels [12]. Using the *optimization transfer* techniques described in [13], nonquadratic regularization can also be included.

2.1. Toeplitz embedding

Usually the weighting matrix is diagonal, *i.e.*, $\boldsymbol{W} = \mathsf{diag}\{w_i\}$. In these cases, the *Gram matrix* $\boldsymbol{A}'\boldsymbol{W}\boldsymbol{A}$ associated with the norm term in (11) is *block Toeplitz* with *Toeplitz blocks*, and has elements

$$[\boldsymbol{A}'\boldsymbol{W}\boldsymbol{A}]_{kj} = \sum_{i=1}^{M} w_i \left|B(\vec{\nu}_i)\right|^2 \mathrm{e}^{-\imath 2\pi\vec{\nu}_i\cdot(\vec{r}_j - \vec{r}_k)}\,. \qquad (14)$$

By defining the Toeplitz matrix $\boldsymbol{T} = \boldsymbol{A}'\boldsymbol{W}\boldsymbol{A}$ and the vector $\boldsymbol{b} = \boldsymbol{A}'\boldsymbol{W}\boldsymbol{y}$, we can rewrite the gradient expression (13) as:

$$\nabla\Psi(\boldsymbol{x}) = \boldsymbol{T}\boldsymbol{x} - \boldsymbol{b} + \nabla\,\mathsf{R}(\boldsymbol{x})\,. \qquad (15)$$

The elements of $\boldsymbol{b} \in \mathbb{C}^N$ are given by

$$b_j = [\boldsymbol{A}'\boldsymbol{W}\boldsymbol{y}]_j = \sum_{i=1}^{M} w_i y_i \,\mathrm{e}^{-\imath 2\pi\vec{\nu}_i\cdot\vec{r}_j}\,, \qquad j = 1, \ldots, N.$$

We can precompute $\boldsymbol{b}$ prior to iterating using an (adjoint) NUFFT operation [8]. This calculation is similar to the gridding reconstruction method. Each gradient calculation requires multiplying the $N \times N$ block Toeplitz matrix $\boldsymbol{T}$ by the current guess of $\boldsymbol{x}$. That operation can be performed efficiently by *embedding* $\boldsymbol{T}$ into a $2^2N \times 2^2N$ *block circulant* matrix and applying a 2-dimensional FFT [14]. This is called the *ACT method* in the band-limited signal interpolation literature [15,16], and it has also been applied to MR image reconstruction both with [17,18] and without [19] an over-sampled FFT. The first row of the circulant matrix is constructed by using 2^{2-1} (adjoint) NUFFT calls to evaluate columns of (14).

Using the gradient expression (13) requires two NUFFT operations per iteration. Each NUFFT requires an over-sampled FFT and frequency-domain interpolations. In contrast, by using the Toeplitz approach (15), each iteration requires two (double-sized) FFT operations. No interpolations are needed except in the precomputing phase of building $\boldsymbol{T}$ and $\boldsymbol{b}$. For an accurate NUFFT, usually we oversample the FFT by a factor of two (in each dimension). Thus, the NUFFT approach and the Toeplitz approach require exactly the same amount of FFT effort, but the NUFFT approach has the disadvantage of also requiring interpolations. The only apparent drawback of the Toeplitz approach (15) is that it "squares the condition number" of the problem so may be less numerically stable. However, in most applications the measurement noise will dominate the numerical noise, and to control measurement noise one will need to include suitable regularization which will also reduce the condition number.

Circulant preconditioners can accelerate the convergence rate of the CG algorithm for such Toeplitz problems [14, 20].

3. ISSUES

In the conjugate phase reconstruction method (3), the sampling density compensation factors $\{w_i\}$ are essential and must be chosen carefully. One advantage of iterative reconstruction methods is that density compensation is not required. In fact, because the gaussian noise in MRI, the Gauss-Markov theorem from statistical estimation theory implies that one should choose $\boldsymbol{W} = \boldsymbol{I}$, the identity matrix, in the cost function (11). Using any other choice may lead to increased noise of the LS estimator. However, in the MR literature several papers have reported using $\boldsymbol{W} = \mathsf{diag}\{w_i\}$ for LS image reconstruction. The purported benefits of this practice are "faster

convergence" and "better conditioning." Often the CG algorithm is initialized with an all zero image, and in that case the first iteration of CG (for the unregularized case) yields $\boldsymbol{x}^{(n)}1 = \boldsymbol{A}'\boldsymbol{W}\boldsymbol{y}$. Using $\boldsymbol{W}$ with density compensation will usually improve the quality of this first iterate. However, that is a weak rationale for *continuing* to use $\boldsymbol{W}$ even after the first iteration. We recommend using $\boldsymbol{W}$ for the first iteration only, or equivalently, initializing CG with the CP reconstruction, and then using $\boldsymbol{W} = \boldsymbol{I}$ thereafter. To improve the conditioning of the problem, we recommend using regularization instead of manipulating $\boldsymbol{W}$, again because of the Gauss-Markov theorem.

Although the presentation in this summary has been for a single receive coil, iterative methods generalize readily to the case of parallel imaging. The matrix $\boldsymbol{A}'\boldsymbol{A}$ in the case of parallel imaging again is Toeplitz, facilitating algorithm acceleration.

4. EXAMPLE

At the workshop we will illustrate the application of iterative image reconstruction methods to non-Cartesian MR, including for the challenging case of under-sampled k-space data. Special regularization methods can help with under-sampled data, *e.g.*, [21].

Matlab software for iterative MR image reconstruction is available [22].

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